UDC 539.3

ON EXACT SOLUTIONS OF THE STATIC PROBLEM OF COMPLEX SHEAR

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An exact solution of the problem of complex shear of a half-space containing an angular groove, is obtained for the strain-hardening law given by (1.1). The latter relation was used earlier to analyze the state in the neighborhood of the weak concentrators, within the framework of the approximate, semi-inverse method of V. V. Sokolovskii [1]. Other examples of the problems of grooves which admit closed solutions are shown, and the method of P-analytic representations of solutions given in [2] is used. Applications to similar problems under different laws of strain-hardening are given in [3].

1. Let us adopt the relations of the deformation theory without unloading, with a single curve

$$\tau = G\gamma / \sqrt{1 + a^2\gamma^2} \tag{1.1}$$

The basic functions, connected with the transformations of the solution equations of the static antiplane problem to the generalized Cauchy — Riemann system [2], can be written as follows:

$$\eta = \frac{1}{2} \ln \frac{\sqrt{1 + a^2\gamma^2} - 1}{\sqrt{1 + a^2\gamma^2} + 1}, \quad p = \sqrt{\frac{\tau'\gamma}{\tau}} = -\operatorname{th} \eta \tag{1.2}$$

$$\frac{\tau}{\tau_0} = \frac{1}{\operatorname{ch} \eta}, \quad \frac{\gamma}{\gamma_0} = -\frac{1}{\operatorname{sh} \eta} \quad \left(\eta < 0;\ \tau_0 = \frac{G}{a}, \quad \gamma_0 = \frac{1}{a}\right)$$

Relations (1.2) and subsequent expressions use the notation of [2]; the integration constant in the first formula is chosen in accordance with the condition that $\eta \to 0$ as $\gamma \to \infty$.

The characteristic of the generalized analytic function $f(\zeta) = \alpha + i\beta$ has the form

$$P(\eta) = \frac{1}{p} \exp\left(\int \left(\frac{1}{p} - p\right) d\eta\right) = \operatorname{cth}^2 \eta$$

therefore the general solution of the system can be written in the form of a linear combination of the derivatives of the harmonic function $u(\xi, \eta)$ [4]

$$\beta = \frac{\partial u}{\partial \eta} - \operatorname{cth} \eta \frac{\partial^2 u}{\partial \eta^2}, \quad \alpha = -\frac{\partial u}{\partial \xi} + \operatorname{th} \eta \frac{\partial^2 u}{\partial \xi \partial \eta} \tag{1.3}$$

$$\beta(\xi, \eta) = (x \cos \xi - y \sin \xi) \gamma / \gamma_0 \tag{1.4}$$

$$\alpha(\xi, \eta) = (x \sin \xi + y \cos \xi) \tau / \tau_0$$

$$(\gamma_{yz} + i\gamma_{xz} = \gamma e^{i\xi})$$

The representation (1.3) enables us to construct, in quadratures, a number of new solutions of the canonical problems dealing with polygonal grooves (in particular the problems of cracks), by reducing them to boundary value problems for the analytic function $\Phi(\zeta) = \partial u / \partial\eta - i\partial u / \partial\xi$ of the complex variable $\zeta = \eta + i\xi$.

2. Let a half-space weakened by an angular notch of depth l, extending to the boundary, be subjected to a homogeneous shear of magnitude γ_∞ at infinity (Fig. 1). Using the relations (1. 2) and (1. 4) we can show that the mapping of the region in the x, y-plane onto the η, ξ-plane becomes a half-strip ($|\xi| \leqslant \pi/2 - \psi$, $\eta < 0$) with a cut along the ray ($\eta < \eta_\infty$, $\xi = 0$). The corresponding points are denoted in Figs. 1 and 2 by the same letters. The quantity η_∞ can be obtained from the first formula of (1. 2) by setting $\gamma = \gamma_\infty$.

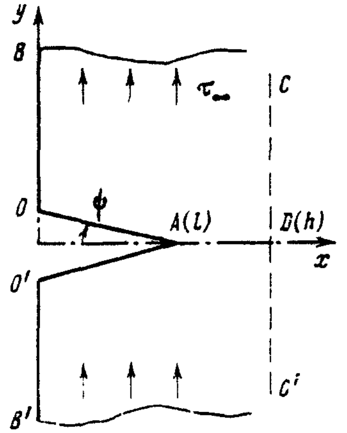

Fig. 1

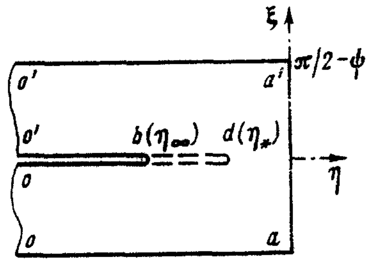

Fig. 2

In obtaining the boundary conditions we note that

$$\xi = 0, \quad x = 0 \quad (OB,\ O'B') \tag{2.1}$$

$$\xi = \pm(\pi/2 - \psi), \quad (x - l)\sin\psi \mp y\cos\psi = 0 \quad (O'A),\ (OA)$$

therefore, in accordance with (1. 2) — (1. 4), we obtain

$$\beta = \frac{\partial u}{\partial\eta} - \operatorname{cth}\eta\,\frac{\partial^2 u}{\partial\eta^2} = \begin{cases} 0, & (ob,\ o'b) \\ -l\sin\psi/\operatorname{sh}\eta, & (o'a',\ oa) \end{cases} \tag{2.2}$$

Integrating the ordinary differential equations with respect to the function $\partial u / \partial\eta$ and taking into account the condition of its boundedness when $\eta \to -\infty$ which follows obviously from (1. 4), we obtain ($\operatorname{Re}\Phi = \partial u / \partial\eta$)

$$\operatorname{Re}\Phi = 0 \quad (ob,\ o'b), \quad \operatorname{Re}\Phi = le^{\eta}\sin\psi \ (oa,\ o'a') \tag{2.3}$$

For any $\xi \in [\psi - \pi/2,\ \pi/2 - \psi]$ and $\eta \to 0$, the asymptotics of the functions have, in accordance with (1. 2) and (1. 4), the form

$$\beta \sim -l\cos\xi / \operatorname{sh}\eta, \quad \alpha \sim l\sin\xi / \operatorname{ch}\eta$$

Using the expressions (1. 3) we can show, that the function $u(\xi, \eta)$ has the following form near the segment aa':

$$u = l\,(c \operatorname{sh} \eta + \operatorname{ch} \eta) \cos \xi, \quad c = \text{const}$$

therefore we can write the boundary condition on aa' as follows:

$$\operatorname{Im} \Phi = -\partial u / \partial \xi = l \sin \xi \quad (\eta = 0, \; |\xi| \leqslant \pi/2 - \psi) \tag{2.4}$$

Moreover we have

$$\Phi(\zeta) \to 0 \quad (\zeta \to \infty) \tag{2.5}$$

To solve the linear boundary value problem (2.3) — (2.5), we pass to the plane of an auxilliary variable

$$\omega = \omega_1 + i\omega_2 = \exp(\zeta / \mu) \quad (\mu = 1 - 2\psi / \pi; \; 0 < \mu \leqslant 1)$$

The region of analyticity of the function $\Phi(\omega) = \Phi(\zeta(\omega))$ is represented by the right semicircle $(|\omega| < 1, \; \omega_1 > 0)$ with a cut along a segment of the real axis $(0 \leqslant \omega_1 \leqslant \omega_\infty)$, with $\omega_\infty = \exp(\eta_\infty / \mu)$. We introduce the following analytic function:

$$\Psi(\omega) = \Phi(\omega) + l\omega^\mu \quad (\arg \omega^\mu = \mu\pi / 2, \; \omega = i) \tag{2.6}$$

and supply it with an additional definition outside the region indicated, as follows:

$$\Psi(\omega) = \overline{\Psi(1/\bar{\omega})}, \quad \Psi(\omega) = -\overline{\Psi(-\bar{\omega})} \tag{2.7}$$

The conditions at the discontinuities have, according to (2.3) and (2.4), the form

$$\Psi_+ - \Psi_- = 0 \quad \text{on} \quad \Gamma_1 \; (|\omega| = 1), \quad \Gamma_2 \; (\omega_1 = 0) \tag{2.8}$$

$$\Psi_+ + \Psi_- = 2g(\omega_1) \quad \text{on} \quad \Gamma_3 \; (\omega_2 = 0, \; |\omega_1| < \omega_\infty, \; |\omega_1| > 1/\omega_\infty)$$

and the function

$$g(\omega_1) = -l\,|\omega_1|^\mu \quad (0 < \omega_1 < \omega_\infty) \tag{2.9}$$

is defined on the remaining segments of the line Γ_3 according to the conditions (2.7) (e.g. $g = l / |\omega_1|^\mu$ when $\omega_1 < -1 / \omega_\infty$). The limit values of the function $\Psi(\omega)$ in the upper half-plane are denoted in the last condition of (2.8) by the subscript plus.

The first two boundary conditions of the conjugation problem (2.8) extend the solutions analytically across the lines Γ_1 and Γ_2.

The canonical function of the problem is

$$X(\omega) = \omega / [(\omega^2 - \omega_\infty^2)(1 / \omega_\infty^2 - \omega^2)]^{1/2} \tag{2.10}$$

$$(X(\omega) = \overline{X(-\bar{\omega})}, \; X(\omega) = \overline{X(1/\bar{\omega})})$$

and we choose the branch of the root for which $\arg X = -\pi / 2$ at the upper edge of the cut $(|\omega_1| < \omega_\infty)$. From the relations (2.8) we obtain

$$\frac{\Psi_+}{X_+} - \frac{\Psi_-}{X_-} = 0 \quad (\Gamma_1, \Gamma_2), \quad \frac{\Psi_+}{X_+} - \frac{\Psi_-}{X_-} = \frac{2g}{X_+} \quad (\Gamma_3) \tag{2.11}$$

Taking into account the symmetry conditions (2.7), (2.8) and the asymptotics (2.5), we can write the solution of the problem (2.11) in the form of a Cauchy type integral

$$\frac{\Psi(\omega)}{X(\omega)} = \frac{1}{2\pi i} \int_{\Gamma_3} \frac{2g(t)\,dt}{X_+(t)\,(t - \omega)}$$

Transforming the latter, with the odd function $g(t)$ taken into account, we obtain

$$\Psi(\omega) = \frac{2}{\pi} X(\omega)\,\omega(\omega^2+1)\int_0^{\omega_\infty} \frac{g(t)(t^2-1)\,dt}{|X_+(t)|(t^2-\omega^2)(t^2\omega^2-1)} \qquad (2.12)$$

and this, together with the formulas (2.6), (2.9) and (1.2) – (1.4), completes the solution of the problem. In particular, the value $\psi = 0$ $(\mu = 1)$ yields the case of a crack-cut emerging at the boundary of the half-plane.

Omitting the calculations for brevity, we shall show certain other formulations of the antiplane problems of solid concentration, admitting the solutions in closed form.

3. In the case of a half-strip with a cut bounded by a stress-free plane $x = h$ (the trace CC' is shown in Fig. 1 with a dashed line), the boundary condition on the additional segment bd of the cut in the image plane can be determined in the same manner as (2.3) from the equation $\beta = -h / \operatorname{sh}\eta$, so that

$$\operatorname{Re}\Phi = h(\operatorname{sh}\eta - \operatorname{th}\eta_*\operatorname{ch}\eta) \quad (\eta_\infty < \eta < \eta_*,\ \xi = 0) \qquad (3.1)$$

The solution of the problem (2.3), (2.4), (3.1) is obtained in the same way as that in Sect. 2, although in this case the value of η_* corresponding to shear deformation at that point D is not known in advance and has to be determined from the condition (2.5). The solution obtained corresponds, at the same time, by virtue of the symmetry, to the case of a plane, weakened along the real axis by a system of rhombic notches or by notches separated from each other by the distance of $2h$.

The cases of planes with periodic or doubly peridoic systems of notches distributed along the direction of the shear, do not present any complications either. Using the conditions of symmetry of the solutions relative to the coordinate lines $y = 2Hn$ $(n = 0, \pm 1, \pm 2, \ldots)$ or $x = 2hk$, $y = 2Hn$ $(k, n = 0, \pm 1, \pm 2, \ldots)$ respectively, we see that the domain of analyticity of the function $\Phi(\zeta)$ is represendted, as before, by a strip with a cut. The boundary conditions (2.4) remain in force; the auxilliary condition at the cut bd changes for the singly periodic system from (3.1) to

$$\operatorname{Im}\Phi = H(\operatorname{ch}\eta - C\operatorname{sh}\eta) \quad (\eta_{**} < \eta < \eta_\infty,\ \xi = 0) \qquad (3.2)$$

The values of the coordinate η_{**}, point b and constant C are obtained from the conditions (2.5) and the fact that the solution is bounded when $\eta \to \eta_{**}$.

For a doubly periodic system of notches, the relation (3.2) must be supplemented by the boundary condition (3.1) for the reintroduced part of the cut de $(\eta_\infty < \eta < \eta_*,\ \xi = 0)$, and the constant η_* has the same meaning as in (3.1).

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